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Regulars

Mathematical mysteries: Zeno's Paradoxes

by Rachel Thomas

The paradoxes of the philosopher Zeno, born approximately 490 BC in southern Italy, have puzzled mathematicians, scientists and philosophers for millennia. Although none of his work survives today, over 40 paradoxes are attributed to him which appeared in a book he wrote as a defense of the philosophies of his teacher Parmenides. Parmenides believed in *monism*, that reality was a single, constant, unchanging thing that he called *'Being'*. In defending this radical belief, Zeno fashioned 40 arguments to show that change (motion) and plurality are impossible.

The most famous of Zeno's arguments is the **Achilles**:

> *'The slower when running will never be overtaken by the quicker; for that which is pursuing must first reach the point from which that which is fleeing started, so that the slower must necessarily always be some distance ahead.'*

This is usually put in the context of a race between Achilles (the legendary Greek warrior) and the Tortoise. Achilles gives the Tortoise a head start of, say 10 m, since he runs at 10 ms^{-1} and the Tortoise moves at only 1 ms^{-1}. Then by the time Achilles has reached the point where the Tortoise started (T_0 = 10 m), the slow but steady individual will have moved on 1 m to T_1 = 11 m. When Achilles reaches T_1, the labouring Tortoise will have moved on 0.1 m (to T_2 = 11.1 m). When Achilles reaches T_2, the Tortoise will still be ahead by 0.01 m, and so on. Each time Achilles reaches the point where the Tortoise was, the cunning reptile will always have moved a little way ahead.

This seems very peculiar. We know that Achilles should pass the Tortoise after 1.11 seconds when they have both run just over 11 m, so Achilles will win any race longer than 11.11m. But why in Zeno's argument does it seem that Achilles will never catch the tortoise?

If you think of the distances Achilles has to travel, first 10 m to T_0, then 1 m to T_1, then 0.1 m to T_2 etc., we can write it as a sum of a geometric series:

$$10 + 1 + 0.1 + ... + 10^{(2-n)} + ...$$

Now it is a little clearer. As the distance that Achilles travels to catch the tortoise is the sum of a geometric series where the multiplier is less than one (read more), we know that the distance is finite (and equal to 11.11m) as the series converges. And as he only has to travel a finite distance, Achilles will obviously cover that distance in a finite time if he is traveling at a constant speed.

So how did Zeno manage to confuse us? Zeno's argument is based on the assumption that you can infinitely divide space (the race track) and time (how long it takes to run). By dividing the race track into an infinite number of pieces, Zeno's argument turned the race into an infinite number of steps that seemed as if they would never end. However, each step is decreasing, and so dividing space and therefore time into smaller and smaller pieces implies that the passage of time is 'slowing down' and can never reach the moment where Achilles passes the Tortoise. We know that time doesn't slow down in this way. The assumption that space (and time) is infinitely divisible is wrong (more on the physical implications of the limiting process).

There are ways to rephrase the Achilles argument that can take our brains in a slightly different direction. In one example, known as Thomson's Lamp, we suspend our disbelief once again and consider a lamp with a switch that we press to turn on, and press again to turn off.

Now, the lamp is initially off and I switch it on. After 1 minute I switch it off. After half a minute I switch it back on. After a quarter of a minute I switch it off. After one eighth of a minute I switch is back on and so on, each time halving the length of time I wait before I switch the lamp on or off as appropriate (I have very quick reflexes). After 2 minutes, (the sum of the infinite series 1 + 1/2 + 1/4 + ...), I will have finished this infinite sequence of actions. So at this point, is the lamp on or off? And will it have made a difference if the lamp was initially on rather than off?

As with Zeno's original version of Achilles, these arguments are based on the infinite divisibility of time, and the paradox that results can be seen to illustrating that time is not infinitely divisible in this way.

Interestingly, as mentioned above, the Achilles paradox was only one of 40 arguments Zeno is thought to have produced, and in another of his arguments called the Arrow, Zeno also shows that the assumption that the universe consists of finite, indivisible elements is apparently incorrect. So, here is where the real paradox of Zeno lies. In his arguments, he manages to show that the universe can neither be continuous (infinitely divisible) nor discrete (discontinuous, that is made up of finite,indivisible parts).

This seeming contradiction in the nature of reality is echoed by concepts from an area developed over 2000 years after Zeno lived, the Theory of Relativity. For example, light is now thought of as having a dual nature, behaving sometimes as a particle or photon (discrete), and at other times like a wave (continuous). In fact even Zeno's belief in monism − in a static, unchanging reality − which was the basis for his producing the arguments in the first place, seems oddly similar to cosmologists ideas about 'worldlines' (the 'history' of a particle in spacetime) where 'the entire history of each worldline already exists as a completed entity in the plenum of space time' (read more).

So Zeno's paradoxes still challenge our understanding of space and time, and these ancient arguments have surprising resonance with some of the most modern concepts in science.

References

Zeno and the Paradox of Motion
Zeno's race course, part 2 − Lecture notes from the University of Washington
Zeno at St Andrews site

About the author

Rachel Thomas is an assistant editor of *Plus*.

Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.